Statistics I
Chapter 3 : Expected values

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Motivation

To describe an experiment (a random experiment), we oftenly use measures that allow us to summarize the available data. The "mean" is one of that measures, since it helps us to localize the center of the distribution. For example:

Example

Students that enroll the postgraduate in Finance have to complete 4 curricular units 2 of them with 10 ECTS and the remaining 2 with 5 ECTS. João has completed the postgraduate in Finance with the following marks:

- 15 - course with 10 ECTS
- 13 - course with 10 ECTS
- 16 - course with 5 ECTS
- 14 - course with 5 ECTS

The average final grade of João was 14, since

$$\frac{15 \times 10 + 13 \times 10 + 16 \times 5 + 14 \times 5}{30} = 14.(3)$$

Discrete random variables

Let X be a a discrete random variable and let D_X be the set of discontinuity points of the cumulative distribution function X. For generality, let us assume that the number of elements of D_X is countably infinite, that this $D_X = \{x_1, x_2, ...\}$.
The probability function of X is given by

$$f_X(x) = \begin{cases} P(X = x) & , x \in D_X \\ 0 & , x \notin D_X \end{cases}$$

Expected Value of a discrete random variable: The expected value of a random variable, denoted as $E(X)$ or μ_X, also known as its population mean, is the weighted average of its possible values, the weights being the probabilities attached to the values

$$\mu_X = E(X) = \sum_{x \in D_X} x \times f_X(x) = \sum_{i=1}^{\infty} x_i \times f_X(x_i).$$

provided that $\sum_{x \in D_X} |x| \times f_X(x) = \sum_{i=1}^{\infty} |x_i| \times f_X(x_i) < +\infty.$

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Example

Let X be a discrete random variable such that

$$X = \begin{cases} 1, & \text{if there is a success} \\ 0, & \text{otherwise} \end{cases} \text{ and } P(X = x) = \begin{cases} p, & \text{if } x = 1 \\ 1 - p, & \text{if } x = 0 \end{cases}.$$

The expected value of X is given by

$$E(X) = p \times 1 + (1 - p) \times 0 = p$$

Remarks:

- If the number of elements of D_x is finite that this
  $D_X = \{x_1, x_2, ..., x_k\}$ where k is a finite integer, then
  $\sum_{x \in D_X} |x| \times f_X(x) = \sum_{i=1}^{k} |x_i| \times f_X(x_i)$ and the condition
  $\sum_{i=1}^{k} |x_i| \times f_X(x_i) < +\infty$ is always satisfied.
- Note that μ_X can take values that are not in D_X.

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Example: Let X be the random variable with the probability function

$$P(X = 2^n) = 2^{-n}, \quad \text{for all } n \in \mathbb{N}.$$

This means that $P(X = x) = 0$ for all $x \notin \{2^n : n \in \mathbb{N}\}$. One can notice that the probability function satisfies the conditions:

1) $P(X = x) \geq 0$ for all $x \in \mathbb{R}$;

2) $\sum_x P(X = x) = \sum_{n=1}^{\infty} 2^{-n} = \frac{1/2}{1-1/2}$

Additionally, one may easily notice that, the distribution does not have expected value:

$$E(X) = \sum_{n=1}^{\infty} 2^n \times 2^{-n} = \infty$$

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Expected values (or mean or expectation) of a continuous random variable: If X is a continuous random variable and $f_X(x)$ is its probability density function at x, the expected value of X is

$$\mu_X = E(X) = \int_{-\infty}^{+\infty} x f_X(x) dx$$

provided that $\int_{-\infty}^{+\infty} |x| f_X(x) dx < \infty$.

Remark: Thus, the mean can be thought of as the centre of the distribution and, as such, it describes its location. Consequently, the mean is considered as a _measure of location_.

Example

Let X be a random variable with probability density function given by

$$f_X(x) = \begin{cases} \frac{1}{b-a}, & a < x < b \\ 0, & \text{otherwise} \end{cases}.$$

Then, the expected value of X is
$E(X) = \int_{-\infty}^{+\infty} x f_X(x) dx = \int_a^b \frac{x}{b-a} dx = \frac{b+a}{2}$

Expected value of a function of a discrete random variable: If X is a discrete random variable and $f_X(x)$ is the value of its probability function at x, the expected value of $g(X)$ is

$$E[g(X)] = \sum_{x \in D_X} g(x) \times f_X(x) = \sum_{i=1}^{\infty} g(x_i) \times f_X(x_i).$$

provided that $\sum_{x \in D_X} |g(x)| \times f_X(x) = \sum_{i=1}^{\infty} |g(x_i)| \times f_X(x_i) < +\infty$.

Expected value of a function of a continuous random variable: If X is a continuous random variable and $f_X(x)$ is its probability density function at x, the expected value of $g(X)$ is

$$E(g(X)) = \int_{-\infty}^{+\infty} g(x)f(x)dx.$$

provided that $\int_{-\infty}^{+\infty} |g(x)| f_X(x)dx < \infty$.

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Remarks:

- The existence of $E(X)$ does not imply the existence of $E(g(X))$ and the inverse is also true.
- $E(g(X))$ can be calculated using the above definition or finding the distribution of $Y = g(X)$ and computing directly $E(Y)$.

Example: Let X be a discrete random variable with probability function given by $f_X(x) = 1/3$, $x = -1,\ 0,\ 1$ and $Y = g(X) = X^2$. We can compute $E(X^2)$ if the following two ways:

- By using the definition of expected value of a function of a random variable

$$E(X^2) = (-1)^2 f_X(-1) + (0)^2 f_X(0) + (1)^2 f_X(1)$$
$$= 1 \times 1/3 + 0 \times 1/3 + 1 \times 1/3 = 2/3.$$

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- Finding the distribution of Y and afterwards computing the expected value of Y.

$$f_Y(0) = P(Y = 0) = P(X = 0) = 1/3$$
$$f_Y(1) = P(Y = 1) = P(X = -1) + P(X = 1) = 1/3 + 1/3 = 2/3.$$

Therefore,

$$f_Y(y) = \begin{cases} 1/3, & y = 0 \\ 2/3, & y = 1 \\ 0, & \text{otherwise} \end{cases}.$$

Thus,
$$E(X^2) = E(Y) = 0 \times f_Y(0) + 1 \times f_Y(1) = 0 \times 1/3 + 2 \times 1/3 = 2/3.$$

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Example: Let X be a random variable such that

$$f_X(x) = \begin{cases} \frac{1}{b-a}, & a < x < b \\ 0, & \text{otherwise} \end{cases}.$$

Then, the expected value of $Y = 2X$ is

$$E(2X) = \int_{-\infty}^{+\infty} 2x f_X(x) dx = \int_{a}^{b} 2\frac{x}{b-a} dx = 2\int_{a}^{b} \frac{x}{b-a} dx = b + a.$$

A different approach to calculate $E(Y)$ is to derive the distribution of Y and afterwards to compute $E(Y)$.

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Indeed,

$$F_Y(y) = P(Y \leq y) = P(2X \leq y) = P(X \leq y/2) = F_X(y/2).$$

The density function can be obtaining differentiating the cdf F_Y:

$$f_Y(y) = F'_Y(y) = (F_X(y/2))' = \frac{1}{2}F'_X(y/2) = \frac{1}{2}f_X(y/2)$$

$$= \begin{cases} \frac{1}{2(b-a)}, & 2a < y < 2b \\ 0, & \text{otherwise} \end{cases}$$

Therefore,

$$E(Y) = \int_{-\infty}^{+\infty} y f_Y(y) dy = \int_{2a}^{2b} \frac{y}{2(b-a)} dy = b + a$$

Properties of the expected value

The expected values satisfy the following properties:

- $E(a + bX) = a + bE(X)$, where a and b are constants.
- $E(X - \mu_X) = E(X) - \mu_X = 0$.
- If a is a constant, $E(a) = a$.
- If b is a constant, $E(b \times g(X)) = bE(g(X))$.
- Given n functions $u_i(X)$ $i = 1, ..., n$ and ,
  $E\left[\sum_{i=1}^{n} u_i(X)\right] = \sum_{i=1}^{n} E[u_i(X)]$.

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Moments of a discrete random variable: The rth moment of a discrete random variable (or its distribution), denoted as μ'_r, is the expected value of X^r

$$\mu'_r = E(X^r) = \sum_{x \in D_X} x^r \times f_X(x) = \sum_{i=1}^{\infty} x_i^r \times f_X(x_i), \text{ for } r = 1, 2, ...$$

provided that $\sum_{x \in D_X} |x|^r \times f_X(x) = \sum_{i=1}^{\infty} |x_i|^r \times f_X(x_i) < +\infty$.

Moments of a continuous random variable: The rth moment of a continuous random variable (or its distribution), denoted as μ'_r, is the expected value of X^r :

$$\mu'_r = E(X^r) = \int_{-\infty}^{+\infty} x^r f_X(x) dx$$

provided that $\int_{-\infty}^{+\infty} |x|^r f_X(x) dx < \infty$.

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r^{th} **central moment of the random variable or** r^{th} **moment of a random variable about its mean**

Central moments of a discrete random variable: The rth central moment of a discrete random variable (or its distribution), denoted as μ_r, is the expected value of $(X - \mu_X)^r$

$$\mu_r = E\left[(X - \mu_X)^r\right] = \sum_{x \in D_X} (x - \mu_X)^r \times f_X(x) = \sum_{i=1}^{\infty} (x_i - \mu_X)^r \times f_X(x_i),$$

provided that
$\sum_{x \in D_X} |x - \mu_X|^r \times f_X(x) = \sum_{i=1}^{\infty} |x_i - \mu_X|^r \times f_X(x_i) < +\infty.$

Central moments of a continuous random variable: The rth central moment of a continuous random variable (or its distribution), denoted as μ_r, is the expected value of $(X - \mu_X)^r$:

$$\mu_r = E\left[(X - \mu_X)^r\right] = \int_{-\infty}^{+\infty} (x - \mu_X)^r f_X(x) dx$$

provided that $\int_{-\infty}^{+\infty} |x - \mu_X|^r f_X(x) dx < \infty.$

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Example

Let X be a random variable such that $f_X(x) = \frac{1}{b-a}$ if $a < x < b$, and $f_X(x) = 0$ if $x \notin (a, b)$.
The 2nd moment is

$$\int_{-\infty}^{+\infty} x^2 f_X(x) dx = \int_a^b \frac{x^2}{b-a} dx = \frac{1}{3}(a^2 + ab + b^2).$$

The 2nd central moment is

$$\int_{-\infty}^{+\infty} (x - \mu_X)^2 f_X(x) dx = \int_a^b \frac{(x - \mu_X)^2}{b-a} dx = \frac{(b-a)^2}{12}.$$

1. μ_1 is of no interest because is it zero when it exists.
2. μ_2 is an important measure and is called variance.
3. μ_3 and μ_4 are also important.

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Variance: the second central moment about the mean of a random variable (μ_2), also called variance, is an indicator of the dispersion of the values of X about the mean.

The variance of a discrete random variable:

$$Var\left(X\right) = \sigma_X^2 = \mu_2 = E\left[\left(X - \mu_X\right)^2\right] = \sum_{x \in D_X} \left(x - \mu_X\right)^2 \times f_X\left(x\right),$$

provided that $Var\left(X\right) < +\infty$.

The variance of a continuous random variable:

$$Var\left(X\right) = \sigma_X^2 = \mu_2 = E\left[\left(X - \mu_X\right)^2\right] = \int_{-\infty}^{+\infty} \left(x - \mu_X\right)^2 f_X(x)dx,$$

provided that $Var\left(X\right) < +\infty$.

Remark: We can show that if $\mu_2' = E\left(X^2\right)$ exists, then both μ_X and σ_X^2 exist.

Properties of the Variance:

- $Var\left(X\right) \geq 0$.
- $\sigma_X^2 = Var\left(X\right) = E\left(X^2\right) - \mu_X^2$.
- If c is a constant, $Var\left(c\right) = 0$.
- If a and b are constants, $Var\left(a + bX\right) = b^2 Var\left(X\right)$.

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Example

Let X be a discrete random variable such that

$$X = \begin{cases} 1, & \text{if there is a success} \\ 0, & \text{otherwise} \end{cases} \text{ and } P(X = x) = \begin{cases} p, & \text{if } x = 1 \\ 1 - p, & \text{if } x = 0 \end{cases}.$$

The expected value of X is given by

$$E(X) = p \times 1 + (1 - p) \times 0 = p$$
$$E(X^2) = p \times 1^2 + (1 - p) \times 0^2 = p$$
$$Var(X) = E(X^2) - (E(X))^2 = p(1 - p)$$

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Example

Let X be a continuous random variable such that

$$f_X(x) = \begin{cases} 2x, & 0 < x < 1 \\ 0, & \text{otherwise} \end{cases}.$$

Then,

$$E(X) = \int_{-\infty}^{+\infty} x f_X(x) dx = 2/3$$

and

$$E(X^2) = \int_{-\infty}^{+\infty} x^2 f_X(x) dx = 1/2.$$

Therefore,

$$Var(X) = E(X^2) - (E(X))^2 = \frac{1}{18}$$

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Standard deviation: The variance is not measured in the scale of the random variable as it is computed using the square function, in order to obtain a measure of dispersion about the mean which is measure in the same scale of the random variable we need to compute the standard deviation. The *Standard deviation* is given by:

$$\sigma_X = \sqrt{Var(X)}.$$

Coefficient of variation: If we are interested in a measure of dispersion which is independent of the scale of the random variable we should use the coefficient of variation. The *coefficient of variation* is given by

$$CV(X) = \frac{\sigma_X}{\mu_X}.$$

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Example

Let X be a discrete random variable such that

$$X = \begin{cases} 1, & \text{if there is a success} \\ 0, & \text{otherwise} \end{cases} \quad \text{and} \quad P(X = x) = \begin{cases} p, & \text{if } x = 1 \\ 1 - p, & \text{if } x = 0 \end{cases}.$$

The expected value of X is given by

$$E(X) = p \times 1 + (1 - p) \times 0 = p$$
$$E(X^2) = p \times 1^2 + (1 - p) \times 0^2 = p$$
$$Var(X) = E(X^2) - (E(X))^2 = p(1 - p)$$

Additionally,

$$\sigma_X = \sqrt{p(1 - p)} \quad \text{and} \quad CV(X) = \sqrt{\frac{1 - p}{p}}$$

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Example

Let X be a continuous random variable such that

$$f_X(x) = \begin{cases} 2x, & 0 < x < 1 \\ 0, & \text{otherwise} \end{cases}.$$

We have already computed

$$E(X) = 2/3 \quad \text{and} \quad Var(X) = \frac{1}{18}.$$

Therefore,

$$\sigma_X = \sqrt{\frac{1}{18}} = \frac{1}{3\sqrt{2}}$$

and

$$CV(X) = \frac{\sigma_X}{\mu_X} = \frac{1}{2\sqrt{2}}.$$

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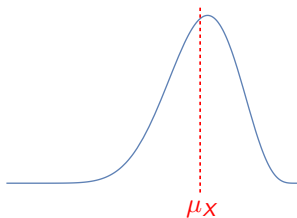Expected values
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Skewness:

Beyond the location and dispersion it is desirable to know the distribution behaviour about the mean. One parameter of interest is the coefficient of asymmetry also known as skewness. This parameter is a measure of asymmetry of a probability function/density about the mean of the random variable. It is given by

$$\gamma_1 = \frac{E\left[(X - \mu_X)^3\right]}{Var\left(X\right)^{3/2}} = \frac{\mu_3}{\sigma_X^3}$$
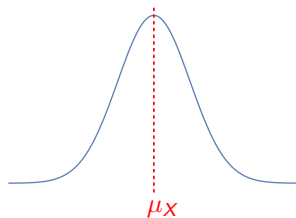
Remarks:

- For discrete random variables a probability function is symmetric if $f_X(\mu_x - \delta) = f_X(\mu_x + \delta)$ for all $\delta \in \mathbb{R}$.
- For continuous random variables the probability density function is symmetric if $f_X(\mu_x - \delta) = f_X(\mu_x + \delta)$ for all $\delta \in \mathbb{R}$
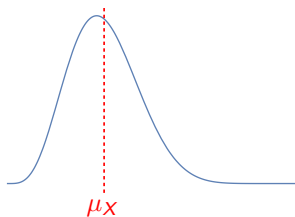
Negative Skew

Zero Skew

μ_X

μ_X

Positive Skew

μ_X

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Example

Let X be a discrete random variable with probability function given by

$$f_X(x) = \left\{ \begin{array}{ll} 0.25 & \text{, for } x = -1 \\ 0.5 & \text{, for } x = 0 \\ 0.25 & \text{, for } x = 1 \end{array} \right. .$$

Note that

$$\mu_X = E(X) = (-1) \times 0.25 + (0) \times 0.5 + 1 \times 0.25 = 0$$

and

$$E(X^3) = (-1)^3 \times 0.25 + (0)^3 \times 0.5 + 1^3 \times 0.25. = 0.$$

Therefore, $f_X(x)$ is symmetric about $\mu_X = 0$ and $\gamma_1 = 0$.

<u>Remark</u>: Note however that we can have $\gamma_1 = 0$, and the probability function/density is not symmetric about the mean, that is $\gamma_1 = 0$ does not imply symmetry.

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Example

Let X be a discrete random variable with probability function given by

$$f_X(x) = \begin{cases} 0.1 & \text{, for } x = -3 \\ 0.5 & \text{, for } x = -1 \\ 0.4 & \text{, for } x = 2 \end{cases}.$$

Note that

$$\mu_X = E(X) = (-3) \times 0.1 + (-1) \times 0.5 + 2 \times 0.4 = 0.$$

Since $f_X(-1) \neq f_X(1)$, this function is not symmetric around $\mu_X = 0$. However

$$E(X^3) = (-3)^3 \times 0.1 + (-1)^3 \times 0.5 + 2^3 \times 0.4. = 0,$$
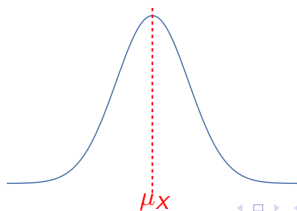
and consequently $\gamma_1 = 0$.

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Kurtosis:

The kurtosis measures the "thickness" of the "tails" of the probability function/density or, equivalently, the "flattening" of the probability function/density in the central zone of the distribution.

$$\gamma_2 = \frac{E\left[(X - \mu_X)^4\right]}{Var\left(X\right)^2} = \frac{\mu_4}{\sigma_X^4}.$$
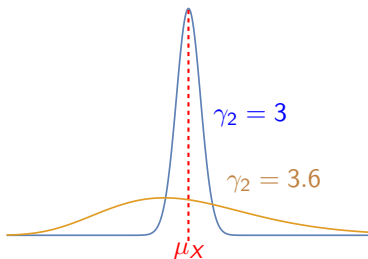
The kurtosis of the normal distribution is 3. We use this distribution as reference, therefore we can define the *excess of kurtosis* as

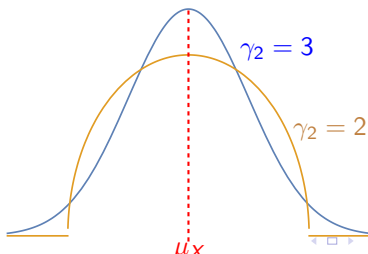$$\gamma_2' = \frac{\mu_4}{\sigma_X^4} - 3$$

Figura: Density of a normal distribution with mean 0 and variance 1.

$$\mu_X$$

Positive excess of kurtosis

$\gamma_2 = 3$

$\gamma_2 = 3.6$

μ_x

Negative excess of kurtosis

$\gamma_2 = 3$

$\gamma_2 = 2$

μ_x

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Quantiles: Other parameters of interest are the quantiles of a (cumulative) distribution or quantiles of a random variable. Quantiles have the advantage that they exist even for random variables that do not have moments.

Definition: Let be X be random variable and $\alpha \in (0, 1)$. The quantile of order α, q_α is the <u>smallest</u> value among all points x in $\mathbb{R}$ that satisfy the condition

$$F_X(x) \geq \alpha.$$

Remarks:

- If X is a discrete random variable $q_\alpha \in D_X$.

- The quantile 0.5 is called the *median* of a (cumulative) the distribution function. It can also be interpreted as a centre of the distribution and therefore it is also considered a measure of location.

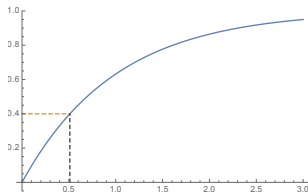- When the probability function/ density is symmetric the *median = mean*.

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Example: Let X be a random variable such that

$$F_X(x) = \begin{cases} 0, & x < 0 \\ 1 - e^{-x}, & xx \geq 0 \end{cases}$$

What is the quantile of order 0.4?
Solution: We can start by solving

$$F_X(x) = 0.4 \Leftrightarrow x = -ln(0.6) \approx 0.5108$$

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Let X be a discrete random variable such that

$$X = \begin{cases} 1, & \text{if there is a success} \\ 0, & \text{otherwise} \end{cases} \text{ and } P(X = x) = \begin{cases} p, & \text{if } x = 1 \\ 1 - p, & \text{if } x = 0 \end{cases}.$$
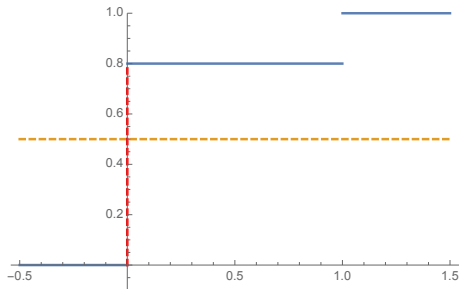
It follows that

$$F_X(x) = \begin{cases} 0 & \text{, for } x < 0 \\ 1 - p & \text{for } 0 \leq x < 1 \\ 1 & \text{, for } x \geq 1 \end{cases}$$

1) Compute the quantile of order 0.5 when $p = 0.2$.
2) Compute the quantile of order 0.5 when $p = 0.6$.

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Fix $p = 0.2$:

$q_{0.5}$ which is the _smallest_ value among all points x in $\mathbb{R}$ that satisfy the condition
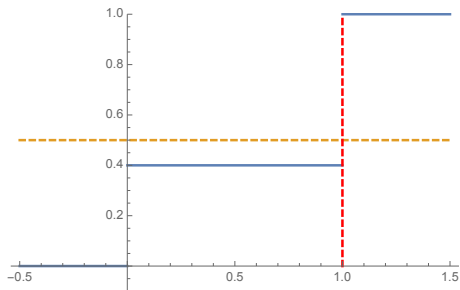
$$F_X(x) \geq 0.5.$$

Therefore, $q_{0.5} = 0$.

Fix $p = 0.6$:

$q_{0.5}$ which is the _smallest_ value among all points x in $\mathbb{R}$ that satisfy the condition

$$F_X(x) \geq 0.5.$$

Therefore, $q_{0.5} = 1$.

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Remarks:

- The q_α are called quartiles if $\alpha = 0.25$, 0.5, 0.75. Therefore the first quartile is $q_{0.25}$, the second quartile is $q_{0.5}$ and the third quartile is $q_{0.75}$

- The q_α are called deciles if $\alpha = 0.1$, 0.2,...,0.9. Therefore the first decile is $q_{0.1}$, the second decile is $q_{0.2}$, etc..

- The q_α are called percentiles if $\alpha = 0.01$, 0.02,. . .,0.99. Therefore the first percentile is $q_{0.01}$, the second percentile is $q_{0.02}$, etc.

- The interquartile range $IQR = q_{0.75} - q_{0.25}$ is considered a measure of dispersion.

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The mode: The mode of a random variable X or distribution is the value $mo(X)$ that satisfies the condition $f_X(mo(X)) \geq f_X(x)$, for all $x \in \mathbb{R}$, where $f_X(x)$ is the probability function in the case of discrete random variables and it is the probability density function in the case of continuous random variables.

Remarks:

1. The mode can also be interpreted as a centre of the distribution and therefore it is also considered a measure of location.

2. In the case of discrete random variable the mode is the most frequent value.

3. The mode does not have to be unique.

4. If the variable probability distribution/density is symmetric and has only one mode, then the mode equals the median and the mean.

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Example

Let X be a discrete random variable such that

$$X = \begin{cases} 1, & \text{if there is a success} \\ 0, & \text{otherwise} \end{cases} \text{ and } P(X = x) = \begin{cases} p, & \text{if } x = 1 \\ 1 - p, & \text{if } x = 0 \end{cases}.$$

Fix $p = 0.2$:

It follows that $mo(X) = \arg\max_{x \in \mathbb{R}} P(X = x) = 0$.

Fix $p = 0.6$:

It follows that $mo(X) = \arg\max_{x \in \mathbb{R}} P(X = x) = 1$.

Fix $p = 0.5$:

It follows that $mo(X) = \arg\max_{x \in \mathbb{R}} P(X = x) = 0$ and 1.

In this case, there are two modes $mo(X) = 0$ and $mo(x) = 1$.

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Example

Let X be a continuous random variable with density function

$$f_X(x) = \left\{ \begin{array}{cl} 0 & \text{, for } x < 0 \\ e^{-x} & \text{for } x \geq 0 \end{array} \right.$$

Compute the mode.
Solution: Since $f_X(x) = 0$ for $x < 0$ and it is a decreasing function for $x \geq 0$, the mode is given by $mo(X) = 0$.

Example

Let X be a continuous random variable with density function

$$f_X(x) = \left\{ \begin{array}{cl} 2x, & 0 < x < 1 \\ 0, & \text{otherwise} \end{array} \right.$$

There is no mode because the density function does not have a maximum.

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The moment generating function: The moment generating function is an important function in probability because it defines uniquely the distribution function (when the moment generating function is properly defined).

Definition: The moment generating function of a
<u>discrete random variable</u> is given by

$$M_X(t) = E\left(e^{tX}\right) = \sum_{x \in D_X} e^{tx} \times f_X(x) = \sum_{i=1}^{\infty} e^{tx_i} \times f_X(x_i),$$

provided that it is finite.

Definition: The moment generating function of a
<u>continuous random variable</u> is given by

$$M_X(t) = E\left(e^{tX}\right) = \int_{-\infty}^{+\infty} e^{tx} f_X(x) dx..$$

provided that it is finite.

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Remarks on the moment generating function (m.g.f.):

- The m.g.f. may not exist.
- If X is a discrete random variables and D_X is finite, then there is always a m.g.f.;
- The moment generating function is a function of t not X;
- If there is a m.g.f., then there are moments of every order. The reverse is not true.
- A distribution which has no moments – or has only the first k moments – does not have a m.g.f..
- The moment generating function is used to calculate the moments.
- The m.g.f. uniquely determines the distribution function. That is, if two random variables have the same m.g.f., then the cumulative distribution functions of the random variables coincide, except perhaps at a finite number of points.

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Theorem: Let X be a random variable with moment generating function M_X defined. Then,

$$\left. \frac{d^r M_X(t)}{dt^r} \right|_{t=0} = \mu_r' = E\left[X^r\right], \ r = 1, 2, 3, \ldots$$

Remark: This result allows us to compute the raw moment of X of order k by computing the kth derivative and evaluate it at the point 0.

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Example: Let X be a random variable such that

$$f_X(x) = \begin{cases} 0.2, & x = 0, 3 \\ 0.5, & x = 1 \\ 0.1, & x = 2 \\ 0, & \text{otherwise} \end{cases}$$

By definition,

$$E(X) = \sum_{x=0}^{3} x f_X(x) = 0 \times 0.2 + 1 \times 0.5 + 2 \times 0.1 + 3 \times 0.2 = 1.3.$$

By using the moment generating function, we have

$$M_X(t) = \sum_{x=0}^{3} e^{tx} f_X(x) = 0.2(1 + e^{3t}) + 0.5e^t + 0.1e^{2t}.$$

Therefore,

$$M_X'(t) = 0.6e^{3t} + 0.5e^t + 0.2e^{2t},$$

and, consequently,

$$E(X) = M_X'(0) = 1.3.$$

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Example

Let X be a discrete random variable such that

$$X = \begin{cases} 1, & \text{if there is a success} \\ 0, & \text{otherwise} \end{cases} \text{ and } P(X = x) = \begin{cases} p, & \text{if } x = 1 \\ 1 - p, & \text{if } x = 0 \end{cases}.$$

The moment generating function $M_X(t)$ is given by

$$M_X(t) = E(e^{tX}) = (1 - p)e^{0 \times t} + pe^{t \times 1} = (1 - p) + pe^t.$$

The derivative of $M_X(t)$ in t is given by

$$\frac{\partial}{\partial t} M_X(t) = pe^t.$$

Therefore,

$$E(X) = \frac{\partial}{\partial t} M_X(t)\Big|_{t=0} = p$$

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Example: Let X be a continuous random variable with density function

$$f_X(x) = \left\{ \begin{array}{cl} 0 & , \text{ for } x < 0 \\ \lambda e^{-\lambda x} & \text{ for } x \geq 0 \end{array} \right.$$

$$
\begin{aligned}
M_X(t) &= E\left(e^{tX}\right) = \int_0^{+\infty} \lambda e^{(t-\lambda)x} dx \\
&= \lambda \lim_{z \to \infty} \int_0^z e^{(t-\lambda)x} dx = \lambda \lim_{z \to \infty} \left[\frac{e^{(t-\lambda)x}}{t-\lambda} \right]_{x=0}^{x=z} \\
&= -\frac{\lambda}{t-\lambda},
\end{aligned}
$$

provided that $t < \lambda$. Now

$$\frac{dM_X(t)}{dt} = \frac{d\left(-\frac{\lambda}{t-\lambda}\right)}{dt} = \frac{\lambda}{(t-\lambda)^2}$$

and, consequently,

$$E(X) = \left. \frac{dM_X(t)}{dt} \right|_{t=0} = \frac{1}{\lambda}.$$

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Proposition: Let X be a random variable such that M_X is its moment generating function. For $a, b \in \mathbb{R}$,

$$M_{bX+a}(t) = E\left[e^{(bX+a)t}\right] = e^{at} M_X(bt).$$

Proposition: Let X_i, with $i = 1, \cdot, n$ be independent random variables such that its moment generating function is given by M_{X_i}. The moment generating function of a sum of independent random variables $S_n = \sum_{i=1}^{n} X_i$ equals the product of their m.g.f.(s).

$$M_{S_n}(t) = M_{X_1}(t) \times M_{X_2}(t) \times ... \times M_{X_n}(t)$$

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